Chapter 1: Introduction & Definitions

Calculus: Given a function $y=f(x)$, the derivative $\frac{dy}{dx}=f'(x)$ is itself a function of x, and is found by some appropriate rule.

$$f(x)=\ln(x+x^2)\ ;\quad f'(x)=\frac{1}{x+x^2}(1+2x).$$

This course: Given an equation such as $\frac{dy}{dx}=2xy$, find a function $y(x)$ that satisfies this equation.

> **Def.: Differential Equation (DE)**: an equation containing the derivatives of one or more dependent variables, wrt one or more independent variables.

Studying the natural world often involves statements or relations involving rates at which one variable, say y, changes wrt another variable, say t. These relations usually take the form of equations containing y and its derivatives $y', y'', \dots, y^{(n)}$ wrt t.

Note: Often denote $\frac{dy}{dt}, \frac{d^2y}{dt^2}, \dots, \frac{d^ny}{dt^n}$

instead of $y, y'', \dots, y^{(n)}$ (don't get stuck on one particular notation, or letter for the independent variable $\rightsquigarrow$ usu. x or t)

$$\frac{d^{n} y}{d t^{n}}$$

order of the derivative

dependent variable

independent variable

Careful: $\frac{d^2y}{dt^2}$ means $y''(t)$. / $\left(\frac{dy}{dt}\right)^2$ means $\left(y'(t)\right)^2$

Applications of DEs:

- Airplane and ship design; controlling flight of ships & rockets
- Earthquake detection & prediction; weather forecasting
- Modeling behavior of nerve cells; modeling disease spread
- Modeling economic systems; pricing financial derivatives

$\vdots$

Common links? They all deal w/ systems that evolve in time.
DE's are the mathematical apparatus to study such systems.

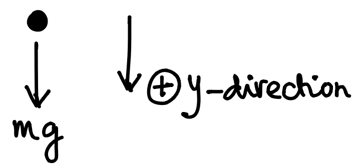
Example: Newton's Second Law of Motion.

Statement: For an object of constant mass m, the sum of the forces acting on the object equals $m \times$ acceleration:

$$F = ma = m \cdot \frac{dv}{dt} = m \cdot \frac{d^2y}{dt^2} \quad \left(v = \frac{dy}{dt},\ y = \text{displacement}\right)$$

This is a differential equation! (involves derivatives of the unknown function $y(t)$, or $v(t)$).

*Gravitational Force:



Free-falling object under the influence of gravity:

$F = mg$ (only force acting on it, $g = 9.8\,m/s^2$ gravitational const.)

$$F = mg \Rightarrow m\frac{d^2y}{dt^2} = mg \Rightarrow \frac{d^2y}{dt^2} = g$$

Integrate: $\dfrac{dy}{dt} = gt + C_1$ (**)

again: $\boxed{y = g\dfrac{t^2}{2} + C_1 t + C_2}$ (*)

<u>Remark</u>: Infinitely many solutions parametrized by C_1, C_2.
To uniquely specify the solution: must augment DE with initial conditions that specify the initial position & velocity of the object: $y(0) = y_0 = 3$, for example

$\dfrac{dy}{dt}(0) = v_0 = 4$, for example

Set $t=0$ in (*): $y(0) = C_2 \Rightarrow C_2 = 3$

Set $t=0$ in (**): $\dfrac{dy}{dt}(0) = C_1 \Rightarrow C_1 = 4$

$$\Rightarrow \boxed{y = g\frac{t^2}{2} + 4t + 3}$$

Initial Value Problem (IVP)

Simplest Kind of DE: $\boxed{\frac{dy}{dt} = g(t)}$ where g is a continuous function

(dependent variable doesn't appear on the right hand side)

→ Solved by direct integration.

Ex: $\frac{dy}{dt} = 1 + e^{2t}$

$$y = \int (1+e^{2t})\,dt = \boxed{t + \frac{1}{2}e^{2t} + C}$$

Ex: $\frac{dy}{dx} = \sin x$

$$y = \int \sin x\,dx = \boxed{-\cos x + C}$$

CLASSIFICATIONS OF DE'S:

1 By Type:

Ordinary Differential Equations (ODE)

- contains only ordinary derivatives of one or more dependent variables wrt a single independent variable:

$$\frac{du}{dx} - \frac{dv}{dx} = x; \quad \frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 6t = 0$$

- main focus of this course

Partial Differential Equations (PDE)

- two or more independent variables involved:

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}; \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}$$

② **By Order**: The order of a DE := the order of the highest-order derivative

- $\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$ Second-order ODE

- $(y-x)dx + 4x\,dy = 0$

  Divide by differential dx: $y - x + 4x\frac{dy}{dx} = 0$

  First order ODE

- $4\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0$ Fourth order PDE

③ **By Linearity**: A linear ODE is one that can be written in the form:

$$\boxed{a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)} \quad (1)$$

Characteristics:

① The dependent variable y and all its derivatives are raised at most to the power 1. (Careful, power not order)

② Each coefficient (the $a_i(x)$) depends only on the indp. var. x.

An equation that is not linear is called nonlinear.

$\boxed{x\frac{dy}{dx} + y = 0}$ linear, 1st order homogeneous

$\boxed{y'' - 2xy' + y = 0}$ linear, 2nd order homogeneous

$\boxed{x^3\frac{d^3y}{dx^3} + 3x\frac{dy}{dx} + 5y = e^x}$ linear 3rd order non homogeneous

Non-linear: $\boxed{yy'' - 2y' = x}$ coefficient depends on y

$\boxed{\frac{d^3y}{dx^3} + \underbrace{\left(\frac{dy}{dx}\right)^2}_{\text{power} = 2} + y = 0}$

Linear eqns. like (1) can be
- **homogeneous** if $g(x) \equiv 0$
- **nonhomogeneous** if $g(x) \not\equiv 0$

Another Example:

$$\boxed{\frac{dy}{dx} = \frac{y}{y-x}}$$

- NOT linear in y (fractions)
- Linear in x:

"flip": $\dfrac{dx}{dy} = \dfrac{y-x}{y} = 1 - \dfrac{x}{y}$

$$\boxed{\frac{dx}{dy} + \frac{1}{y}x = 1}$$

ok because, thinking of x as the dependent var. and y as the indp. var, this coefficient $\frac{1}{y}$ depends only on the indp. var.

Preview for next week: Separable Equations $\dfrac{dy}{dx} = \dfrac{g(x)}{h(y)}$

Ex: $\dfrac{dy}{dx} = -\dfrac{x}{y}$ (Cross-multiply)

$y\,dy = -x\,dx$ (Integrate both sides)

$\int y\,dy = -\int x\,dx$

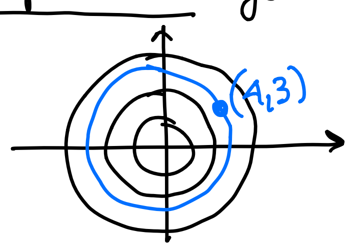
$$\boxed{\frac{y^2}{2} = \frac{-x^2}{2} + C}$$

→ Rewrite: $\dfrac{x^2+y^2}{2} = C \Rightarrow x^2+y^2 = 2C$

the c is generic

$\boxed{x^2+y^2=C}$ → example of an implicit solution

Integral Curves: geometrical representation of general solution



~ Concentric circles centered @ the origin

~ Make it an IVP: $y(4)=3$

$4^2+3^2=C \Rightarrow C=25$ $\boxed{x^2+y^2=25}$